Lec 20: 11/04/2010

White Dwarfs:

We start discussing possible star remnants with the white dwarfs. These are the endpoint of stellar evolution that are supported by degeneracy pressure of the electrons. White dwarfs are formed from the evolution of stars with an initial mass of $\simeq (1-8)\,M_\odot$. The typical densities are in the range $10^5\,g\,cm^{-3}$ – $10^9\,g\,cm^{-3}$. For $\rho \gtrsim 10^6\,g\,cm^{-3}$, the degenerate electron gas is in the relativistic regime.

For $\rho \lesssim 10^6\,g\,cm^{-3}$, the electron gas is in the non-relativistic regime. As we saw, the radius has the following mass dependence in this regime:

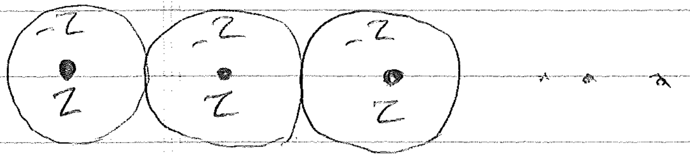
$$R \propto M^{-\frac{1}{3}}$$

This implies that $R \to \infty$ in the limit of $M = 0$, which looks counterintuitive. The point is as M decreases and R increases, the density ρ decreases quickly. The electron gas will not be degenerate anymore at sufficiently low densities. Thus the relationship $R \propto M^{-\frac{1}{3}}$ will not hold anymore, and one has transition to $R \propto M^{\frac{1}{3}}$ for a uniform density distribution.

There are also additional effects at low or high densities that lead to deviations from the simple picture of a gas of degenerate electrons.

Coulomb Corrections at Low Densities:

The Coulomb interactions between the ions and electrons result in a correction to the pressure. To calculate this, we can divide the gas into spherical cells of radius r_0. In each cell there is an ion at the center, surrounded by a uniform electron distribution:



The net charge of each cell is zero, which implies that:

$$r_0 = \left(\frac{3Z}{4\pi n_e}\right)^{\frac{1}{3}}$$ (n_e: number density of free electrons)

The electrostatic potential energy of a cell has two pieces: the potential related to a spherical distribution of electrons, and the potential from interactions between the ion and the surrounding electrons. They are respectively given by:

$$E_{ee} = \frac{3}{5}\frac{Ze^2}{r_0}$$

$$E_{ei} = -\frac{3}{2}\frac{Z^2e^2}{r_0}$$

Hence:

$$E_C = E_{ee} + E_{ei} = -\frac{9}{10}\frac{Z^2e^2}{r_0}$$

The Coulomb energy per electron is:

$$\varepsilon_{(n_e)} = \frac{E_C}{n_e} = -\frac{9}{10}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} Z^{\frac{2}{3}} e^2 n_e^{\frac{1}{3}}$$

The corresponding pressure is found to be:

$$\boxed{P_C = \frac{\partial \varepsilon}{\partial\left(\frac{1}{n_e}\right)} = -\frac{3}{10}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} Z^{\frac{2}{3}} e^2 n_e^{\frac{4}{3}}}$$

The total pressure will then be:

$$P = P_0 + P_C = K\rho^{\frac{5}{3}} - \frac{3}{10}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} Z^{\frac{2}{3}} e^2 n_e^{\frac{4}{3}}$$

$$K = \frac{(3\pi^2)^{\frac{2}{3}}}{5} \frac{\hbar^2}{m_e}\left(\frac{1}{m_u \nu_e}\right)^{\frac{5}{3}}$$

This results in:

$$P = K\rho^{\frac{5}{3}}\left[1 - \left(\frac{\rho_0}{\rho}\right)^{\frac{1}{3}}\right] \qquad \left(\rho_0 \simeq 0.4 Z^2 \nu_e \text{ g cm}^{-3}\right)$$

Equating this to $\frac{GM^2}{R^4}$ (for hydrostatic equilibrium), we obtain the following mass-radius relation:

$$\boxed{\frac{R}{R_\odot} = \left[\left(\frac{M}{M_\odot}\right)^{\frac{1}{3}} + \left(\frac{M_c}{M}\right)^{\frac{1}{3}}\right]^{-1}, \quad M_c = 2.8 \times 10^{-7} Z^2 \nu_e M_\odot}$$

It is seen that $R \propto M^{\frac{1}{3}}$ for $M \lesssim M_{max}$, while $R \propto M^{-\frac{1}{3}}$ for $M \gtrsim M_{max}$, where:

$$M_{max} \equiv (M_0 M_c)^{\frac{1}{2}} \simeq 1.1 \times 10^{-3} Z\left(\frac{\nu}{2}\right) M_\odot$$

Note that M_{max} is close to the planetary masses.

Corrections at High Densities:

s the mass approaches the Chandrasekhar limit, the equation f a state of a degenerate gas will not be valid anymore because of various effects. Here we consider two of the most important such effects.

Neutronization:

As the density increases, the Fermi energy of electrons will rise ($E_F \propto \rho^{1/3}$ in the case of a relativistic degenerate gas). For high enough E_F, the inverse β-decay $e^- + p \to n + \nu_e$ (hence "neutronization") can take place from combination of electrons with protons in nuclei. This reduces the number of electrons, and hence the degeneracy pressure. As a result the system will become unstable.

The threshold energy for neutronization can be estimated as follows. If the Fermi energy is larger than the difference between binding energies of the two nuclei (A,Z) and $(A, Z-1)$, then neutronization will be favoured. The threshold value of the number density is given by:

$$n = \frac{1}{3\pi^2 \lambda_e^3}\left[\left(\frac{Q}{m_e c^2}\right)^2 - 1\right]^{3/2} , \qquad \lambda_e \equiv \frac{\hbar}{m_e c}$$

The most relevant reactions for the white dwarfs are:

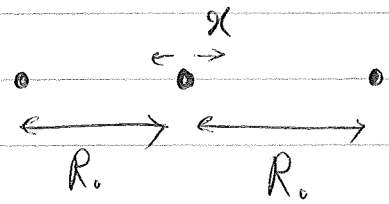
$^4He \rightarrow {}^3H + n \rightarrow 4n$	$\rho \gtrsim 1.37 \times 10^{11}$	$g\,cm^{-3}$
${}^{12}_{6}C \rightarrow {}^{12}_{5}B \rightarrow {}^{12}_{4}Be$	$\rho \gtrsim 3.90 \times 10^{10}$	$g\,cm^{-3}$
${}^{16}_{8}O \rightarrow {}^{16}_{7}N \rightarrow {}^{16}_{6}C$	$\rho \gtrsim 1.90 \times 10^{10}$	$g\,cm^{-3}$

Above these densities, the white dwarfs become unstable as a result of a rapid disappearance of electron pressure. Our description of a degenerate electron gas that supports the white dwarf clearly fails at these densities.

Pycnonuclear Reactions:

Another effect at high density has to do with the zero-point quantum mechanical energy of ions. Because of such a contribution to the kinetic energy, nuclear reactions between two ions can proceed even for zero temperature.

The potential felt by an ion can be approximated by calculating the Coulomb potential between that ion and its nearest neighbours. Considering a one dimensional lattice of ions with separation R_0 between two neighbouring ions, we find;

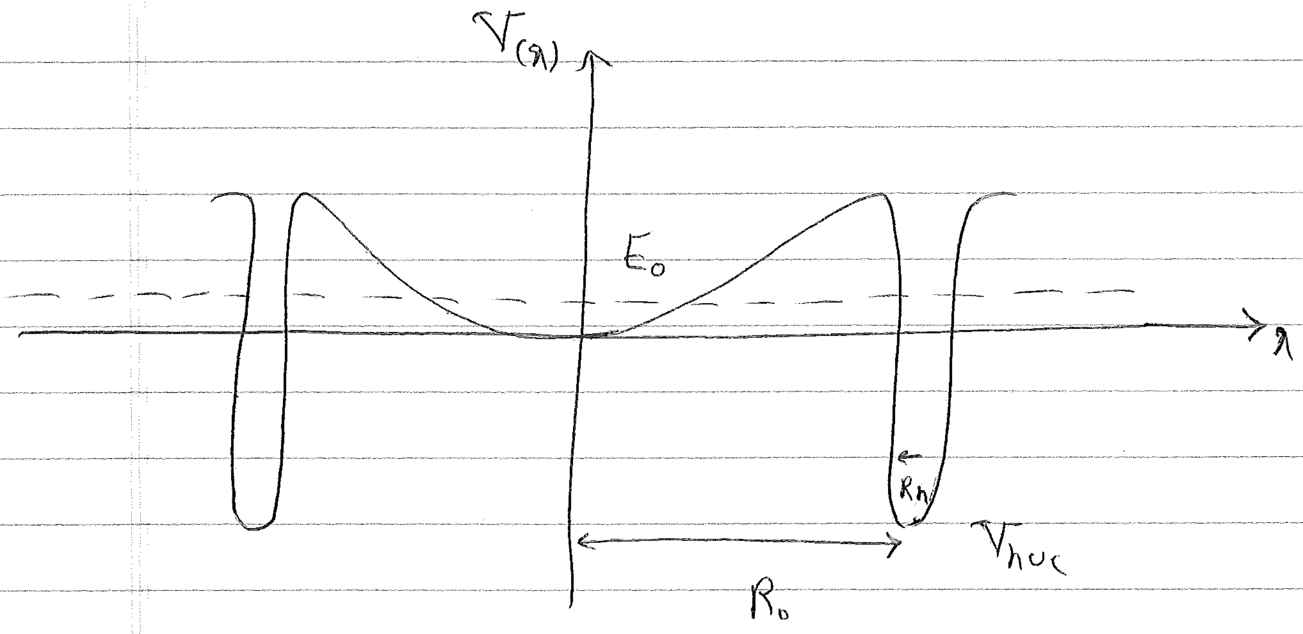


$$V_{(x)} = \frac{Z^2 e^2}{R_0 - x} + \frac{Z^2 e^2}{R_0 + x} \qquad (|x| < R_0 - R_n)$$

Here R_n is the range of the short distance nuclear force. For small $|x|$ we find (up to a constant piece)

$$V_{(x)} \simeq \frac{1}{2} M_A \Omega_0^2 x^2 \quad , \quad \Omega_0^2 = \frac{4 Z^2 e^2}{M_A R_0^3}$$

Here M_A is the mass of ion. This is the potential for a simple harmonic oscillator. To be precise, this potential should smoothly join V_{nuc}, which comes from the attractive nuclear force, for large $|r|$. Pictorially, we have:



The ground state energy of the ion with mass M_A is:

$E_0 = \frac{1}{2} \hbar \Omega_0 \propto R_0^{-3/2}$

The ion is confined to a region of size $r_0 \equiv \left(\frac{2\hbar}{M_A \Omega_0} \right)^{\frac{1}{2}}$, and the wave function (which is a Gaussian) can be approximated as $|\psi|^2 \simeq (\pi^{3/2} r_0^3)^{-1}$ within this region.

The important point to note is that $E_0 \propto R_0^{-3/2}$, and hence it increases with an increase in density.

The tunneling probability for an ion with energy E_0 is given by:

$$T = \exp\left[-\frac{2}{\hbar}\int_{r_0}^{R_0-R_n} dr\,(2M_A V_{(r)} - E_0)^{\frac{1}{2}}\right]$$

Defining:

$$u = \frac{r}{R_0}, \quad \alpha = \frac{E_0 R_0}{2Z^2e^2}$$

We have:

$$T = \exp\left[-2\left(\frac{M_A Z^2 e^2 R_0}{\hbar^2}\right)^{\frac{1}{2}}\int_{\frac{r_0}{R_0}}^{1-\frac{R_n}{R_0}}\left(\frac{u^2}{1-u^2}-\alpha\right)^{\frac{1}{2}} du\right]$$

In the limit $R_n \to 0$ and $r_0 \ll R_0$, which is valid in cases of physical interest, we find:

$$T \simeq \frac{R_0}{r_0}\exp\left(-2\frac{R_0^2}{r_0^2}\right)$$

The reaction rate per ion pair is given by:

$$W = v \left(\pi^{3/2} r_0^3\right)^{-1} \frac{T S(E)}{E} \qquad \left(v = \sqrt{\frac{2E_0}{M_A}}\right)$$

Here $S_{(E)}$ is the astrophysical S factor for the relevant nuclear reaction.

Putting everything together, we get:

$$\boxed{W = 4\left(\frac{2}{\pi^3}\right)^{\frac{1}{2}} S \frac{(Z^2 e^2 M_A)^{3/4}}{(\hbar^2 R_0)^{\frac{5}{4}}} \exp\left[-4Ze \frac{(M_A R_0)^{\frac{1}{2}}}{\hbar}\right]}$$

The number of reactions per unit volume per second is given by:

$$R = n_A W = \left(\frac{\rho}{A}\right) A^2 Z^4 S \gamma \lambda^{\frac{5}{4}} \exp\left(-\varepsilon \lambda^{-\frac{1}{2}}\right)$$

Here A is the atomic weight, $\gamma \simeq 1.1 \times 10^{44}$, $\varepsilon \simeq 2.85$, and:

$$\lambda \equiv \frac{\hbar^2}{2M_A Z^2 e^2} \left(\frac{n_A}{2}\right)^{\frac{1}{3}} = \frac{\hbar^2}{2M_A Z^2 e^2} \left(\frac{3}{\pi}\right)^{\frac{1}{3}} \frac{1}{R_0}$$

To determine the critical densities at which the reactions become important, let us define a time scale through $Rt = n_A$.

For $t = 10^5$ yr, we can solve the equation for R to find ρ if we know the S factor. For example:

$$\boxed{S_{cc} = 8.83 \times 10^{16} \text{ MeV b} \Rightarrow \rho = 6 \times 10^{9} \text{ g cm}^{-3}}$$

This shows that ^{12}C fusion to ^{24}Mg can proceed efficiently for densities above $\sim 10^9 \text{ g cm}^{-3}$. The pycnonuclear reactions can therefore be a major source of instability at white dwarf densities.